TWO NON-ISOMORPHIC SIMPLE GROUPS OF THE SAME ORDER 20,160.*

By Ida May Schottenfels.

1. Introduction. It has never been proved that two simple groups of the same order are necessarily holoedrically isomorphic;† nor has any example been heretofore noticed of two simple groups of the same order which are not holoedrically isomorphic, or abstractly identical.

Dickson‡ has proved that all k-ary linear fractional substitution groups in the Galois Field $[p^n]$§ (p is a prime, n is a positive integer) with determinant unity, are simple; and, among others, he enumerates the ternary linear fractional substitution group of order $8!/2$ in the Galois Field $[2^2]$.

Since no element of period fifteen and no element of period six could be found in this group, the suspicion arose that this group was not holoedrically isomorphic to the alternating group of degree eight,‖ a simple group of the same order, containing substitutions of periods fifteen and six.

At Professor Moore's suggestion the investigation is made as to whether or not these two groups are holoedrically isomorphic, and it is found that *no such isomorphism exists*. The present paper furnishes the first direct proof of this theorem, which has recently been corroborated by Mr. Dickson, whose proof will shortly be published.

2. The Ternary Group. The first group with which we have to deal is the ternary fractional group in the Galois Field $[2^2]$. Without pre-

*This paper was read before the Seminar on Group Theory, held by Professor Moore, March, 1898; and later before the Chicago Mathematical Club held January 21, 1899, at the Univ. of Chicago.

†Holoedric isomorphism is the only isomorphism that can exist between two simple groups.

‡Dickson: Annals of Mathematics, 1st Ser. Vol. 11, 1897, pp. 175-178.

§The expression Galois Field is perhaps not yet in general use. The notion is due to Galois and is fully developed by Serret: *Algèbre supérieure*, Vol. 2, pp. 122-189. The theory in its abstract form is developed by Moore: *Proceedings of the Congress of Mathematics of 1893 at Chicago*, pp. 208-242, 1896, also in the *Bull. of the N. Y. Math. Soc.*, Vol. 3, pp. 73-78, 1894, and by Borel et Drach: *Théorie des Nombres et Algèbre supérieure* 1895, pp. 42-50, 58-62. 343-350.

‖Jordan: *Traité des Substitutions*, pp. 380-382, and Moore: *Math. Ann.* Vol. 51, pp. 417-444, have shown that the quaternary linear homogeneous substitution group of order $8!/2$ in the Galois Field $[2^1]$, and the alternating group of degree eight, both of which are simple, *are* holoedrically isomorphic.

supposing any knowledge of the theory of the Galois Field, this group may be defined as follows:

It is a group of so-called *Fractional Matrices*:

$$\left(\begin{array}{ccc} \xi_{11} & \xi_{12} & \xi_{13} \\ \hline \xi_{21} & \xi_{22} & \xi_{23} \\ \hline \xi_{31} & \xi_{32} & \xi_{33} \end{array}\right) = (\underline{\underline{\xi_{ij}}}). \qquad (i, j = 1, 2, 3)$$

The elements of these matrices are quantities of the form $a + b\rho$ where ρ is a complex cube root of unity and a and b are rational integers. Two such matrices $(\underline{\underline{\xi_{ij}}})$, $(\underline{\underline{\xi'_{ij}}})$ are defined as equal if, and only if, there exists a quantity μ of the form $a + b\rho$ (a, b, ρ having the same meaning as above) such that:*

$$\mu\, \xi_{ij} \equiv \xi'_{ij} \pmod{2}.$$

We consider only those matrices whose determinant is congruent to 1 (mod. 2). *These matrices form a group of order* $8!/2$, the ternary group in question, *provided that we take as our law of composition the general law of matrices.*

$$A = (\underline{\underline{a_{ij}}}), \quad B = (\underline{\underline{b_{jk}}}), \quad C = (\underline{\underline{c_{ik}}}), \quad AB = C, \quad \sum_{j=1,2,3} a_{ij} b_{jk} = c_{ik}.$$

The identity matrix is

$$I = \left(\begin{array}{ccc} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right).$$

We make use of two other special matrices of this group,†

$$\left(\begin{array}{ccc} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \hline 1 & 1 & 0 \end{array}\right) = B, \qquad \left(\begin{array}{ccc} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \hline 1 & \rho & 0 \end{array}\right) = C.$$

The matrices B, C, and $F = CB^6$ satisfy the following system Σ of relations:

$$\Sigma(B, C, F): \quad \left\{ \begin{array}{l} B^7 = C^7 = (B^2C)^5 = (BC)^4 = (CB)^2 = I \\ F = CB^6, \; F^2 = I, \; C = FB \end{array} \right\}.$$

---

* A congruence $a + b\rho \equiv a' + b'\rho$, when a, b, a', b' are rational integers and ρ a complex cube root of unity, is equivalent to the two congruences $a \equiv a'$, $b \equiv b'$.

† These are special cases of a type of generator employed by Moore, *Math. Ann.*, Vol. 51, p. 436.

3. Comparison of the Ternary and Alternating Groups. We must seek to find, in the most general way, in the alternating group $G^8_{8!/2}$, substitutions b, c, and $f=cb^6$ satisfying the system Σ of relations,

$$\Sigma(b, c, f): \qquad \left\{ \begin{array}{l} b^7 = c^7 = (b^2c)^5 = (bc)^4 = (cb^6)^2 = i \\ f = cb^6, f^2 = i, \ c = fb \end{array} \right\} ;$$

and if the ternary group above described be holoedrically isomorphic to the alternating, the alternating *must contain* substitutions b, c, and $f=cb^6$ satisfying the system Σ of relations, when i is the identity substitution of the group.

The alternating group of degree eight is well known, and its substitutions of period seven and of period two can be listed, and the following method of proof will suggest itself.

For b choose any one of the substitutions of period seven, for f any one of period two, and if the product $fb=c$ is of period seven make further trial of b^2c; should this be of period five, try finally the product bc and should this fail to be of period four, reject this f and assume for trial another f, say f' and go through the same process with f', b. Continue in this way until the f's are exhausted, and reject the assumed b. Starting with another b, say b', continue the same process. Finally a solution is found or else the b's must be exhausted. Owing to the very large number of f's and b's, this method is not feasible. In the following a comparatively short method of proof is exhibited.

SYNOPSIS OF THE PROOF.

§4. One substitution only of period seven is necessary for the investigation and the choice is made of $b' = (1\,2\,3\,4\,5\,6\,7)$.

§5. Only forty-five substitutions of period two *distinct with respect to transformation by powers of* $b' = (1\,2\,3\,4\,5\,6\,7)$ are needed in the investigation.

§6. These forty-five substitutions are discovered and listed in the table, page 152.

§7. The table shows wherein these substitutions fail to obey the system Σ of relations and definitive conclusions are drawn from these facts.

PROOF.

4. The most general substitution of period seven in the alternating

group is: $b=(l_1\,l_2\,l_3\,l_4\,l_5\,l_6\,l_7)$ where $l_1\,l_2\,l_3\,l_4\,l_5\,l_6\,l_7$ are seven of the eight numbers 1, 2, 3, 4, 5, 6, 7, 8 taken in any order. The transformed substitution of a substitution π by another substitution s, we denote by π_s so that $\pi_s=s^{-1}\pi\, s$.

Assume that there exists in the alternating group $G^8_{8!/2}$ one set of substitutions (b, c, f) satisfying the system of relations $\Sigma\ (b, c, f)$. By transforming by any substitution t of the symmetric group we have the set (b, c, f) transformed into (b_t, c_t, f_t), and these also are substitutions of the alternating group, since it is invariant under transformation by the symmetric group. Further, the transformed set also satisfy the system of relations $\Sigma\ (b_t, c_t, f_t)$, as shall presently be proved. Finally, the substitution t can be so chosen that b is transformed into any desired substitution b' of period seven, *e. g.* into the particular substitution $b'=(1\ 2\ 3\ 4\ 5\ 6\ 7)$. For, if we have, as above, $b=(l_1\,l_2\,l_3\,l_4\,l_5\,l_6\,l_7)$, then, using the two-rowed notation, we may take as t the substitution

$$t=\begin{pmatrix} l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & l_7 & l_8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix},$$

whence, evidently, we have the desired result:

$$t^{-1}\,b\,t=b_t=(1\ 2\ 3\ 4\ 5\ 6\ 7)=b'.$$

The proof of the second statement is the following: Given two products as in the Σ system of relations, $\Pi_1(b, c, f)=\Pi_2\,(b, c, f)$, then by transforming by t we have $\Pi_1\,(b, c, f)_t=\Pi_2\,(b, c, f)_t$ since this is a group process. Applying the theorem:

The transform of a product is the product of the transforms of the individual factors, and making use of the notation $(b_t, c_t, f_t), =(b', c', f')$, we have

$$\Pi_1\,(b', c', f')=\Pi_2\,(b', c', f').$$

From the elements of period seven, $b'=(1\ 2\ 3\ 4\ 5\ 6\ 7)$ is chosen for the remainder of this discussion.

5. A system of substitutions s' exists, such that

$$b'_{s'}=b'=(1\ 2\ 3\ 4\ 5\ 6\ 7)=(i+1, i+2, \ldots\ i+7)\ \text{modulo}\ 7,\ (i=0, 1, \ldots\ 6)$$

i. e. the transformers s' leave b' invariant. The system of substitutions s' are powers of b'; for, using the two-rowed notation for the substitution s',

$$s'=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ a_0 & a_0+1 & a_0+2 & a_0+3 & a_0+4 & a_0+5 & a_0+6 \end{pmatrix}=b'^{a_0-1}\ (\textit{modulo}\ 7).$$

This fact together with the relation, $c = fb$, reduces the work seven-fold

$$(b', c) f_{b'^i} = (b', c', f') \text{ where } c' = f' b'.$$

Hence f' alone requires investigation, and but one f' from each of the systems of seven, resulting from transformation by powers of b', $b'_{b'^i}$.

6. A detailed study is now made of the substitutions f of the alternating group, where $f^2 = i$. These separate into two divisions (see table page 152). Division I consists of 15 products of four independent transpositions, distinct with respect to transformation by b'^i, which separate into 5 classes of 3 each. Division II consists of 30 products of two independent transpositions distinct with respect to transformation by b'^i, which separate into two sets, Set I, Set II, each containing 15 types. The 15 types of Set II separate into 5 classes of 3 each. These forty-five types have been so arranged in the table that the analytical proof of their distinctness with respect to transformations by powers of b'^i follows very readily.

In Set II, Division II, the 3 substitutions of each of the five classes have the same residual numbers. Aside from 8, the omission of which characterizes them all, the residuals of the five classes respectively are the following:

$$\left.\begin{matrix}1\\2\\3\end{matrix}\right| (156), \quad \left|\begin{matrix}4\\5\\6\end{matrix}\right| (127), \quad \left|\begin{matrix}7\\8\\9\end{matrix}\right| (134), \quad \left|\begin{matrix}10\\11\\12\end{matrix}\right| (146), \quad \left|\begin{matrix}13\\14\\15\end{matrix}\right| (145).$$

A very neat graphical proof of the distinctness of these 5 classes may be made by considering these residuals as the vertices of triangles at the corners of a regular 7-gon, which triangles have, under cyclic permutation of the 7-gon into itself, 7 distinct positions; and by a discussion of the cyclic intervals into which each 7-gon is divided by its respective triangle.

7. The table on the next page shows that no one of the $15 + 30$ f's satisfies the system Σ of relations. We have then proved that in the alternating group there exists no solution (b, c, f) of the system Σ of relations $\Sigma(b, c, f)$.

There exists a totality of elements, BC-products in our ternary group, which have no corresponding elements in the alternating group.

Hence we have here an example of two simple groups of the same order, which are not holoedrically isomorphic, and so an example of two simple abstract groups of the same order which are not identical.

*Table of the 45 f's and of their relation to the System Σ.***

DIVISION I.		DIVISION II.			
SET I.		SET I.		SET II.	
1	(81) (23) (45) (67)*	1	(81) (23)‡	1	(23) (74)*
2	(81) (23) (46) (57)‡	2	(81) (24)*	2	(24) (73)‡
3	(81) (23) (47) (56)*	3	(81) (34)‡	3	(34) (27)*
4	(81) (24) (35) (67)‡	4	(81) (45)‡	4	(45) (36)*
5	(81) (24) (36) (57)*	5	(81) (46)*	5	(46) (35)‡
6	(81) (24) (37) (56)‡	6	(81) (56)‡	6	(65) (34)*
7	(81) (25) (34) (67)*	7	(81) (76)‡	7	(76) (25)*
8	(81) (25) (36) (47)*	8	(81) (72)*	8	(27) (65)*
9	(81) (25) (37) (64)*	9	(81) (26)*	9	(26) (57)‡
10	(81) (26) (34) (75)‡	10	(81) (35)*	10	(35) (27)*
11	(81) (26) (35) (47)*	11	(81) (37)*	11	(37) (25)†
12	(81) (26) (37) (54)‡	12	(81) (57)*	12	(57) (23)*
13	(81) (27) (34) (65)*	13	(81) (25)*	13	(23) (76)*
14	(81) (27) (35) (46)*	14	(81) (36)*	14	(26) (73)†
15	(81) (27) (36) (54)*	15	(81) (47)*	15	(27) (36)*

**In the table the inequalities

$(fb)^7 \neq i \qquad (b^2 c)^5 \neq i \qquad (b c)^4 \neq i$

are denoted respectively by the marks * ‡ †.

CHICAGO, JANUARY, 1900.